VOL.40 Nº1 2023
https://doi.org/10.22491/rmct.v40i1.12121.en

Identification of nonlinear systems by fitting LPV models with polynomial coefficients

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ABSTRACT: *This paper presents a method for the identification or tuning of LPV models with polynomial coefficients. The method is applicable to multivariable systems and to fit the behavior of nonlinear systems. It also presents an extension for multiple varying parameters. A Quarter-Car suspension model was used to illustrate the proposed method, and models with endogenous and exogenous parameters were adjusted.*

KEYWORDS: System identification; Nonlinear systems; LPV systems; Quasi-LPV; LPV model.

RESUMO: Este artigo apresenta um método para a identificação ou ajuste de modelos LPV com coeficientes polinomiais. O método é aplicável a sistemas multivariáveis e para a aproximação do comportamento de sistemas não lineares. Discute-se também a extensão para múltiplos parâmetros variantes. Um modelo de suspensão de um automóvel foi empregado para ilustrar o método proposto, tendo sido ajustados modelos com parâmetros endógenos e exógenos.

PALAVRAS-CHAVE: Identificação de sistema; Sistemas não lineares; Sistemas LPV; Quasi-LPV; Modelos LPV.

1. Introduction

In terms of controlling nonlinear systems, the classic gain scheduling approach, which uses linear design methods and their already consolidated tools, is among the approaches proposed in the literature. Although it is widely used, it does not guarantee stability and performance outside the operating points considered in linear designs, especially in cases where the parametric variation rate is high. The fact that there is a connection between the formulation of Linear Matrix Inequalities (LMIs) and the Lyapunov Theory has allowed the stability and performance criteria developed for linear systems to be extended to non-stationary linear systems, especially for the general class of Linear Parameter Varying (LPV) systems [1].

LPV control, with local or global stability and performance guaranteed in large envelopes of the operating domain of multivariable systems, has been presented as a real alternative to the classic gain scheduling approach. LPV controller gains are automatically programmed without the need for any ad hoc method or interpolation. Since the mid-1990s, LPV control techniques have evolved significantly through three distinct methods [2, 3], namely: polytopic approach, grid-based approach, and Linear Fractional Transformation (LFT). In polytopic approaches, models of some operating points are considered, in principle generated by the extreme values of the coordinates of the vector of varying parameters. The remaining operating points are obtained from the affine combination of these extreme models, which may not be true. The disadvantage of this type of approach is also related to conservatism, which will probably include a range of situations that, despite being considered, may not occur in practice. On the other hand, the optimization problem to be solved, initially of infinite dimension, becomes a problem with finite dimension and equal to the number of vertices of the polytope, since the other models are determined by the affine combination of the models at these vertices [4, 5, 6].

In grid-based approaches, the space of varying parameters is tested in a grid of values, considering realistic trajectories. The advantage of this type of methodology is the reduction of conservatism in relation to the polytopic approach. Recursive and LMI algorithms of this type of methodology are found in the literature. On the other hand, it presents serious restrictions in relation to the number of varying parameters, that is, the computational effort grows exponentially and makes it impossible to treat LPV systems with more than two varying parameters. In addition, it also assumes that the system under analysis is relatively well-behaved, so that its dynamic can be approximated without a large increase in the grid density [7, 8, 6].

In the case of LFT approaches, the LPV model must be transformed into an LFT. Therefore, applications of this methodology are restricted to cases in which the LPV models present specific functions of the varying parameters, such as polynomial functions of the varying parameters. Once the LFT model has been obtained, μ synthesis and others can be used to calculate the controller [9, 10].

A wide variety of LPV control applications were initially developed in the aeronautical area, but applications are expanding to several other areas, and are validated by experiments or high-fidelity simulations, as presented in [11]. One of the main bottlenecks today in the application of LPV control techniques is the lack of methods for obtaining LPV models. This need has promoted the interest of the scientific community working in the area of system identification, in order to be able to produce models of nonlinear or nonstationary systems, with the ultimate objective of using existing LPV control methods.

System identification methods aim to obtain models from the measured signals of the inputs and outputs of a plant under study [12]. Basically, they can be classified into two fronts, depending on the structure of the model: parametric and nonparametric identification. The case of nonparametric identification involves an undetermined structure and, consequently, an a priori undetermined number of parameters. Nonparametric LPV identification has basically been divided into three main approaches: (a) the scattering function; (b) the least-squares support vector machine (LS-SVM), and (c) the one based on the Bayesian configuration, respectively [13, 14, 15] as cited in [16]. Regarding parametric prediction, the structure to be identified is previously established and a certain number of parameters must be adjusted.

The methods for identifying LPV systems [17] can be classified, as defined in [18], into two main areas, according to the mathematical representation used, LPV-IO (Input-Output) and LPV-SS (State Space).

LPV-SS methods adopt a discrete representation in state space or its equivalent LFR, which allows the representation of Multiple-Input Multiple-Output (MIMO) systems. More information about the approaches that use the LPV-SS structure can be found in [18].

This work uses the LPV-IO representation, which utilizes discrete time series models as its mathematical structure. In most cases, it uses the LTI error prediction configuration and is generally treated only for the Single-Input Single-Output (SISO) case. A discrete model in the form of time series can be represented as:

$$y(k) = -\sum_{i=1}^{n} a_i(\theta) q^i y(k) + \sum_{j=0}^{m} b_j(\theta) q^j u(k) + e(k) \qquad (1)$$

where q is the delay operator in the time domain, so that:

$$q^p y(k) = y(k-p) \qquad (2)$$

The variable $e(k)$ is the process noise, usually a white noise with zero mean, $n \geq m$ and the coefficients $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=0}^{m}$are dependent on the parameter θ The prediction of the coefficients a_i and b_j in the model (1) can be performed as defined in [18]:

(i) Interpolation approach. The methods that use this approach are those originating from the classic concept of gain-scheduling, characterized by considering as the operating point specific values of the varying parameter that, once frozen, determines the error prediction structure of the LTI system, allowing the identification of local models. The global model is obtained through the interpolation of local models, as developed in [19, 20].

(ii) Set association approach. In this case, the noise in the measured data is treated as deterministic uncertainty and, instead of a direct estimate of the coefficients, a set of their feasible values is calculated, as presented in [21, 22]. This feasible set represents the values of the coefficients that satisfy the model equation in (1) and a priori with an assumed error less than or equal to that of the noise in the measured data. A direct estimate of the coefficient is obtained by calculating the mean of the values in the feasible set. This approach generally uses non-convex programming methods.

(iii) Nonlinear programming approach. The coefficients $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=0}^{m}$ of the time series model in (1) are estimated using nonlinear programming methods to minimize the mean squared prediction error [13, 23]. The objective is to achieve a better prediction than that of linear regression methods. In some cases, this is done through a non-linear parameterization:

$$a_i(\theta) = \alpha_{i,0} + \alpha_{i,1} Z$$
$$b_j(\theta) = \beta_{j,0} + \beta_{j,1} Z$$

where $\alpha_{i,0}$, $\alpha_{i,1}$, $\beta_{j,0}$ and $\beta_{j,1} \in \mathbb{R}$ and the variable Z is the output of an artificial neural network that uses as inputs the output vector $[(y(k)\ y(k\text{-}1)\ \cdots)]^T$, the input vector $[(u(k)\ u(k\text{-}1)\ \cdots)]^T$ and the vector of the measures of the varying parameter $[(\theta(k)\ \theta(k\text{-}1)\ \cdots)]^T$ of the system to be identified. This approach, in most cases, uses a mixed procedure of linear and non-linear programming through linear regression methods combined with neural networks.

(iv) Linear regression approach. Structures of linear models of discrete time series are used, such as Autoregressive with Exogenous Inputs (ARX), widely used in the literature on the identification of LTI systems, which is part of the system identification package of the MATLAB® programming and numeric computing platform.

In the LPV case, coefficients $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=0}^{m}$ of (1) are polynomial functions of the varying parameter, such as:

$$a_i(\theta) = \alpha_{i,0} + \sum_{p=1}^{N_i} \alpha_{i,p}\theta^p$$

$$b_j(\theta) = \beta_{j,0} + \sum_{p=1}^{M_j} \beta_{j,p}\theta^p$$

In this way, it uses the concept of predicting the LTI error via Least-Squares (LS), recursive or not, as well as instrumental variables that lead to a better adjustment in the presence of noisy signals. As a result, a linear model in the parameters is obtained by linear regression, according to the precursor work of [24], an approach used in this work.

In [25] a method of LPV-IO identification is proposed using the linear regression approach, which seeks a parsimonious, nonparametric model structure that can capture the unknown dependence of the coefficients and as a function of the varying parameter in (1). This dependence can vary between polynomial, rational, or even discontinuous functions. To obtain an efficient solution, the article proposes the LS-SVM method, which leads to a construction of the model without the a priori information of order and delay of the system under study. It was originally developed as a class of supervised learning methods, as presented in [26, 27] as cited in [25], where it is used to obtain the model structure.

In [28], a method based on Instrumental Variable (IV) for bias correction was developed to identify SISO LPV-IO models, of the ARX type, from measurements of the output and the signal of the varying parameter, corrupted by noise. The noise process associated with the output is considered colored, of zero mean and with unknown distribution, while the measurements of the signal of the varying parameter are affected by a white Gaussian noise. The proposed method eliminates the bias resulting from the methods originating from LS when they neglect the measurement noise existing in the signal of the varying parameter. Thus, it provides a consistent estimate of LPV models with polynomial dependence of the varying parameter, whose instrument used only needs to be uncorrelated with the noise that corrupts the output observations. In this way, an approximation for the noise-free varying parameter does not need to be calculated.

The work developed in [29] is an analysis of the one-step-ahead error prediction criterion used in LPV identification processes, with the aim of obtaining new kernel functions [27, 30] to be applied in LPV-IO identification processes of nonparametric Box-Jenkins models.

In [31] a bias correction scheme for closed-loop identification of LPV-IO models is presented, using the regression approach, caused by the correlation between the input signal that excites the process and the output noise. The proposed identification algorithm provides a consistent estimate of the open-loop model parameters when the output signal and the variable signal of the variable parameter are corrupted by measurement noise.

A nonparametric LPV-IO identification by regression is presented in [32] , using LS-SVM to estimate the skid angle of a passenger car, replacing the sensor used in commercial vehicles, due to its high cost. The problem of the article is inspired by [21], which uses the set association approach method, but the article uses the method proposed in [25].

The work of [33] presents a study of LPV-IO identification through the regression approach, in the search for a global model with a nonparametric structure without requiring much a priori information about the model order, using Hilbert space through Reproducing Kernel Hilbert Space (RKHS), which corresponds to a global quadratic optimization problem directly solvable with LMI constraints, for the selection of the structure of the parsimonious model to be identified.

An LPV-IO method by online regression is presented in [34], in which the analysis of the dynamics is performed in the domain of the varying parameter or as a function of it named by the authors as *causal regressor*, instead of the time domain. As a result, the coefficients to be identified are re-estimated using only present data and at least one previous estimate, that is, each coefficient prediction does not necessarily depend on its estimate at the previous instant, but on one of its past estimates at an instant when its associated causal regressors are similar to the present ones. The proposed criterion for measuring this similarity is the norm of the differences between the current and past causal regressors. The proposed method presented low computational cost for similar adjustment of the coefficients, compared to traditional offline methods.

In this work, some ideas presented in [24] were adopted, and the following additional developments were proposed: extension of the method to multivariate systems; obtainment of the solution from a batch of stored data instead of estimate by the recursive form; use of polynomials with independent degrees in each model coefficient; expansion with multiple varying parameters and the realization of the time lag in the model between the current output and the most recent input, which corresponds to the increase in the relative degree of the transfer function of the identified model. The proposed technique was also used to approximate nonlinear models in the Quasi-LPV format by LPV models with polynomial coefficients. To illustrate the proposed method, a car suspension system was explored in several situations, with parameter variations and nonlinearities, approximating them by LPV models.

Regarding the structure of this work, section 2 presents the definitions of LPV and Quasi-LPV systems, section 3 presents the proposed method with the extension to multivariable system and expansion to multiple varying parameters. Section 4 presents the problem addressed for a quarter-car suspension model with several nonlinear situations, depending on the type and quantity of variable parameters. Finally, section 5 presents the final considerations and conclusions.

2. LPV/Quasi-LPV Systems

An LPV or Quasi-LPV system is one whose matrices *A, B, C* and *D* in the state space representation are not constant and vary depending on parameters that are exogenous or endogenous to the system. These parameters, which alter the system dynamic, are called variable parameters θ. An exogenous variable parameter is one that is external to the system; an endogenous parameter is characterized as being internal to the system, represented by one of the states of the system or a function of it. Both variable parameters must be measurable in order to portray the system. Depending on the variable parameters used in the system representation, the models can be classified as LPV or Quasi-LPV.

An LPV model is recognized by having all exogenous and measurable varying parameters. The following is the definition of LPV models.

Definition 1 – LPV Model [35], as cited in [36]: Given a compact subset $P \subset \mathbb{R}^d$, F_p represents the operator that maps $t \in \mathbb{R}^+$ onto a vector $\theta(t)$ of external parameters, whose components are continuous functions by parts $\forall t \in \mathbb{R}^+$ Consider also the continuous functions, $A{:}P \rightarrow \mathbb{R}^{n\times n}$, $B{:}P \rightarrow \mathbb{R}^{n\times w}$, $C{:}P \rightarrow \mathbb{R}^{\zeta\times n}$, and $D{:}P \rightarrow \mathbb{R}^{\zeta\times w}$. An LPV model of order n is defined as:

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}A(\theta(t)) & B(\theta(t))\\ C(\theta(t)) & D(\theta(t))\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}. \quad (3)$$

It can be added that the Linear Time Varying (LTV) model is a particular case of an LPV system, in which the dynamic matrices depend on the varying parameter $\theta(t) = t$.

A Quasi-LPV model is a nonlinear model that resembles the LPV model in (3). In this case, the vector $\theta(t)$ is composed of two types of varying parameters, exogenous and endogenous, both measurable.

Definition 2 – Quasi-LPV Model [36]: Be $\theta(t) \in P$ such that $\theta(t)=[\Omega(t)^T\ z(t)^T\ ]^T$, where $\Omega(t)$ corresponds to the vector of exogenous variables, similarly to the LPV system in (3), and $z(t)$ corresponds to the vector of endogenous variables, containing some states of the system or functions of these, which interfere in the system dynamic. The Quasi-LPV model can be defined by:

$$\begin{bmatrix}\dot{z}(t)\\ \dot{\eta}(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}A_{11}(\theta(t)) & A_{12}(\theta(t)) & B_1(\theta(t))\\ A_{21}(\theta(t)) & A_{22}(\theta(t)) & B_2(\theta(t))\\ C_1(\theta(t)) & C_2(\theta(t)) & D(\theta(t))\end{bmatrix}\begin{bmatrix}z(t)\\ \eta(t)\\ u(t)\end{bmatrix},$$

where the state vector $x(t)=[z(t)^T\,\eta(t)^T\ ]^T$ and $\eta(t)$ represents the vector containing the states that do not interfere in the model matrices.

The following example depicts a mathematical manipulation to transform a nonlinear system into a Quasi-LPV model. The endogenous varying parameters considered are responsible for the nonlinearities of the system and must be measurable to characterize this representation.

According to [2], as an example, consider the nonlinear plant modeled by the following equations:

$$\dot{x}_1 = \mathrm{sen}(x_1) + x_2$$
$$\dot{x}_2 = x_1 x_2 + u$$

where u is the input of the system and considering x_1 and x_2 as the model states, $x = [x_1\ x_2]$ can be defined as the state vector. Thus, a Quasi-LPV representation of this nonlinear model could be

$$\dot{x} = \begin{bmatrix}\mathrm{sen}(x_1)/x_1 & 1\\ x_2 & 0\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u.$$

This representation may not be adequate unless x_1 and x_2 are measurable and $x_1 \neq 0\ \forall t \in \mathbb{R}^+$ In this case, there are only the endogenous varying parameters and $\eta(t) = \emptyset$. Thus $x(t)=z(t)$ and $z(t)=[x_1\ x_2]^T$.

Assuming that one has only x_1 measurable, a more adequate representation could be:

$$\dot{x} = \begin{bmatrix}\mathrm{sen}(x_1)/x_1 & 1\\ 0 & x_1\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u,$$

which allows rewriting the equation so that the state matrix only has dependence on the variable x_1, that is, $z(t) = x_1$ and $\eta(t) = x_2$.

3. Method proposed

The problem addressed here consists of adjusting an LPV model with polynomial coefficients, so that its output $\hat{y}$ approximates, according to some previously defined standard, the output of the nonlinear physical system. The proposal is that the model be discrete and in the form of a time series, whose coefficients can be polynomials dependent on the varying parameter $\theta = \theta(t)$. Initially, we are dealing only with the SISO case with a single varying parameter. To identify the

VOL.40 Nº1 2023
https://doi.org/10.22491/rmct.v40i1.12121.en

model, it is assumed that the system under study is previously monitored with sensors, so that the temporal data of its inputs, outputs and the varying parameter, even if continuous, are acquired according to a convenient sampling rate T.

In the same way as in [24], the class of discrete-time LPV models was adopted, parameterized as follows:

$$A(q,\theta)y(k) = B(q,\theta)u(k) \quad (4)$$

where q is the delay operator, as defined in (2), which leads to polynomials as a function of the varying parameter, according to:

$$A(q,\theta) = 1 + a_1(\theta)q + \cdots + a_n(\theta)q^n \quad (5)$$

$$B(q,\theta) = b_1(\theta)q^r + b_2(\theta)q^{r+1} + \cdots + b_m(\theta)q^{r+m-1} \quad (6)$$

In addition, it was considered that the varying parameter θ, although continuous, was transformed into a function of discrete time, that is, $\theta \coloneqq \theta(kT) = \theta_k$, where T is the sampling period.

Thus, by (4), (5) and (6), the structure of the identified model, in the form of a time series, can be written as:

$$\begin{aligned} y_k &= -a_1 y_{k-1} - a_2 y_{k-2} - \cdots - a_n y_{k-n} + \\ &+ b_1 u_{k-r} + b_2 u_{k-r-1} + \cdots + b_m u_{k-r-m+1}, \end{aligned} \quad (7)$$

where $y_k = y(kT)$ and $u_k = u(kT)$ n, is the order of the model and represents the number of autoregressive terms of the output signal, m is the number of input terms, m is the delay between the current output and the most recent input considered, and $m + r - 1 \leq n$.

It was also considered that the coefficients of the model above have polynomial dependence in relation to the varying parameter $\theta = \theta_k$. Thus, $\forall i \in \{1,\cdots,n\}$ e $N_i \in \mathbb{N}$ and :

$$a_i = a_i(\theta) = \alpha_{i,0} + \alpha_{i,1}\theta + \alpha_{i,2}\theta^2 + \cdots + \alpha_{i,N_i}\theta^{N_i}. \quad (8)$$

Similarly, for the coefficients of the input variable u, $\forall j \in \{1, \cdots, \text{m}\}$ and $M_j \in \mathbb{N}$:

$$b_j = b_j(\theta) = \beta_{j,0} + \beta_{j,1}\theta + \beta_{j,2}\theta^2 + \cdots + \beta_{j,M_j}\theta^{M_j} \quad (9)$$

The varying parameter $\theta(t)$ is considered measurable, but may be out of phase with the current output, that is:

$$y_k = f(\theta_{k-\gamma}), \quad (10)$$

where $\gamma \in \{0,\cdots, n\}$. Usually, is adopted, that is, the determination of the current output depends on the value of θ at the previous instant.

Although the structure in (7) was adopted in the form of a time series, it is important to note that, defined in this way, it will have a one-to-one correspondence with the models in the form of state space, requiring only the use of a canonical realization. In this way, the calculation of eigenvalues of a model can be determined in the usual way, from the characteristic equation:

$$\det\left(A(\theta(t) - \lambda I\right) = 0.$$

Considering the examples in section 4, the vectors containing the orders of the polynomials in each coefficient of the autoregressive terms of the output and of the input terms are defined, as per (8) and (9):

$$\begin{aligned} N &= [N_1 \quad N_2 \quad \cdots \quad N_n] \in \mathbb{N}^n \\ M &= [M_1 \quad M_2 \quad \cdots \quad M_m] \in \mathbb{N}^m \end{aligned} \quad (11)$$

The number ξ of parameters to be identified can be calculated by means of:

$$\xi = n + m + \textstyle\sum_{i=1}^{n} N_i + \sum_{j=1}^{m} M_j. \quad (12)$$

Given the system (4), structured as per (5) to (12), ***Theorem 1*** shows how the polynomial coefficients to be identified are determined, from the resolution of a system of linear equations and the input, output and varying parameter data, all previously measured in the physical system.

Theorem 1 – Consider u_k, y_k and θ_k, with $k \in \{1,\cdots, p\}$, the measured data series referring, respectively, to the input, the output and the varying parameter of the system in (4). The polynomial coefficients in (5) and (6) of the LPV model can be determined by solving the following system of linear equations:

$$AX = \begin{bmatrix} A_\alpha & A_\beta \end{bmatrix} \begin{bmatrix} X_\alpha \\ X_\beta \end{bmatrix} = B, \quad (13)$$

where:

$$A_\alpha = \begin{bmatrix} Y_{n-1,N_1} & Y_{n-2,N_2} & \cdots & Y_{0,N_n} \\ Y_{n,N_1} & Y_{n-1,N_2} & \cdots & Y_{1,N_n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{p-1,N_1} & Y_{p-2,N_2} & \cdots & Y_{p-n,N_n} \end{bmatrix},$$

$$A_\beta = \begin{bmatrix} U_{n-r,M_1} & U_{n-r-1,M_2} & \cdots & U_{n-r-m+1,M_m} \\ U_{n-r+1,M_1} & U_{n-r,M_2} & \cdots & U_{n-r-m+2,M_m} \\ \vdots & \vdots & \ddots & \vdots \\ U_{p-r,M_1} & U_{p-r-1,M_2} & \cdots & U_{p-r-m+1,M_m} \end{bmatrix},$$

$$X_\alpha = -[\alpha_{1,0} \quad \alpha_{1,1} \quad \cdots \quad \alpha_{1,N_1} \quad \alpha_{2,0} \quad \cdots \quad \alpha_{n,N_n}]^T,$$
$$X_\beta = [\beta_{1,0} \quad \beta_{1,1} \quad \cdots \quad \beta_{1,M_1} \quad \beta_{2,0} \quad \cdots \quad \beta_{m,M_m}]^T,$$
$$B = [y_n \quad y_{n+1} \quad y_{n+2} \quad \cdots \quad y_p]^T \in \mathbb{R}^{p-n+1},$$
$$Y_{k,p} = y_k[1 \quad \theta_{k-\gamma} \quad \theta_{k-\gamma}^2 \quad \cdots \quad \theta_{k-\gamma}^p] \in \mathbb{R}^{p+1},$$
$$U_{k,p} = u_k[1 \quad \theta_{k-\gamma} \quad \theta_{k-\gamma}^2 \quad \cdots \quad \theta_{k-\gamma}^p] \in \mathbb{R}^{p+1}.$$

Demonstration: The system of linear equations in (13) comes directly from (7). For a given instant of time $t = kT$, equation (7) can be rewritten with the help of (8) and (9) as follows:

$$y_k = -(\alpha_{1,0} + \cdots + \alpha_{1,N_1}\theta^{N_1})y_{k-1} - \cdots$$
$$-(\alpha_{n,0} + \cdots + \alpha_{n,N_n}\theta^{N_n})y_{k-n} +$$
$$+(\beta_{1,0} + \cdots + \beta_{1,M_1}\theta^{M_1})u_{k-r} + \cdots$$
$$+(\beta_{m,0} + \cdots + \beta_{m,M_m}\theta^{M_m})u_{k-r-m+1},$$

so that:

$$y_k = [Y_{k-1,N_1} \cdots Y_{k-n,N_n} U_{k-r,M_1} \cdots$$
$$\cdots U_{k-r-m+1,M_m}]\begin{bmatrix} X_\alpha \\ X_\beta \end{bmatrix}.$$

Considering that $k \in \{n, n+1, \cdots, p\}$, we arrive at the system of linear equations in (13).

The linear system resulting from ***Theorem 1*** is overdetermined. Several methods can be used to solve it, such as pseudo-inverse, partial scaling and pivoting, Gauss-Jordan and others. An alternative is to use nonlinear programming methods, seeking the minimization of $\|AX - B\|$, which is a convex problem [37]. However, in this case, by means of mathematical transformations, it is also possible to use linear programming methods, such as LMI resolution packages [38].

In the case of a multivariable system with ω inputs and ζ outputs, it is assumed that each of the outputs can be identified independently, that is, that the original problem can be decomposed into a set of ζ Multiple-Input Single-Output (MISO) problems with ω independent inputs. Thus, for each output i, with $i \in \{1,2,\cdots,\zeta\}$

$$\mathrm{A}_i(q,\theta)y_{i,k} = \mathrm{B}_1(q,\theta)u_{1,k} + \cdots + \mathrm{B}_\omega(q,\theta)u_{\omega,k}, \quad (14)$$

Where $y_{i,k}$ represents the output y_i at instant $t = kT$ Using ***Theorem 1*** and developing (14), analogously to the SISO case, it is possible to determine a system of linear equations to calculate the coefficients dependent on the varying parameter θ in A_i, B_1, $\cdots$, B_ω.

With multiple inputs, the vector **M** in (11) is transformed into a matrix, with each row presenting the degrees of the polynomial expansions of the terms of each input. The number of coefficients ξ to be identified in (12), considering a single varying parameter θ, will be:

$$\xi = n + m + \sum_{i=1}^{n} N_i + \sum_{i=1}^{\omega}\sum_{j=1}^{m} M_{i,j}. \quad (15)$$

3.1 Adjustment Indices

The assessment of the adjustment error between the behavior of the model and the real dynamic system is carried out in two parts. In the first part, called the coefficient adjustment error, the error in the calculation of the coefficients during the identification process is tested, based on the data measured at the input and output of the physical system. Using (13) and that $v = p - n + 1$, the adjustment error vector is defined as:

$$\begin{bmatrix} e_{a,1} \\ e_{a,2} \\ \vdots \\ e_{a,v} \end{bmatrix} = \begin{bmatrix} b_1 - a_{11}x_1 - \cdots - a_{1v}x_v \\ b_2 - a_{21}x_1 - \cdots - a_{2v}x_v \\ \vdots \\ b_v - a_{v1}x_1 - \cdots - a_{vv}x_v \end{bmatrix},$$

and the adjustment indexes of the coefficients as:

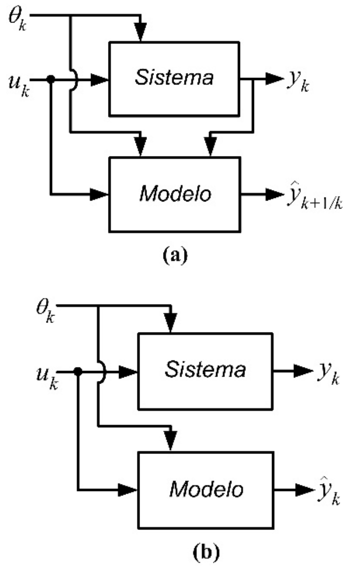
$$J_2^a = \|E_a\|_2 = \sqrt{|e_{a,1}|^2 + |e_{a,2}|^2 + \ldots + |e_{a,v}|^2}, \quad (16)$$

$$J^a_\infty = \|E_a\|_\infty = \max\{|e_{a,1}|,|e_{a,2}|,\dots,|e_{a,v}|\}, \qquad (17)$$

It can be seen that B is the vector containing the measured outputs of the system. In this case, the error for generating the current output is evaluated, considering that the previously measured outputs are available. This corresponds to the one-step-ahead prediction of the output, that is, $\hat{y}_{k+\frac{1}{k}}$, as illustrated in Figure 1(a). It is also equivalent to saying that the model uses the previous output measurements of the system in real time for the prediction. It is important to note that the use of the model in this format should not be confused with recursive prediction, since its coefficients are already determined and will not be adjusted during operation.

Once the model has been identified, it is possible to evaluate the prediction error independently of the system, based on a simulation, which is much more rigorous. A new input signal is adopted for validation, with the same initial conditions for the system and model. In this case, the model is considered autonomous, that is, its output $\hat{y}_k$ is generated exclusively from the input provided and the trajectory of the varying parameter θ without interference from the system output y, as shown in Figure 1(b).

Figure 1 - Simulation of the predicted outputs: (a) one step ahead; (b) independent.



The error for a horizon of h simulation periods is evaluated as follows:

$$E_s = \begin{bmatrix} e_{s,1} \\ e_{s,2} \\ \vdots \\ e_{s,h} \end{bmatrix} = \begin{bmatrix} y_1 - \hat{y}_1 \\ y_2 - \hat{y}_2 \\ \vdots \quad \vdots \\ y_h - \hat{y}_h \end{bmatrix} = y - \hat{y},$$

and the simulation or validation indices by:

$$J^s_2 = \|E_s\|_2 = \sqrt{|e_{s,1}|^2 + |e_{s,2}|^2 + \dots + |e_{s,h}|^2}, \qquad (18)$$

$$J^s_\infty = \|E_s\|_\infty = \max\{|e_{s,1}|,|e_{s,2}|,\dots,|e_{s,h}|\}, \qquad (19)$$

It should be noted that, using the same initial conditions, input signals, sampling rate and sample size in the one-step-ahead simulation and in the free simulation, we obtain $J^a_2 \leq J^s_2$.

3.2 Expansion by Multiple Varying Parameters

In the situation where there are multiple varying parameters $\theta(t) = [\theta_1(t)\ \theta_2(t) \cdots \theta_d(t)]^T$, the expansion of the model coefficients can be performed in analogy with (8) and (9), also considering the cross terms of the varying parameters. Thus, for the case of two varying parameters, the coefficients of the model output terms and, equivalently, for the input ones, would have the following format:

$$\begin{aligned} a_i = a_i(\theta_1,\theta_2) &= \alpha_{i,0} + \alpha_{i,1}\theta_1 + \alpha_{i,2}\theta_1^2 + \cdots + \\ &+ \alpha_{i,N_i}\theta_1^{N_i} + \alpha_{i,N_i+1}\theta_2 + \cdots + \alpha_{i,2N_i}\theta_2^{N_i} + \\ &+ \alpha_{i,2N_i+1}\theta_1\theta_2 + \alpha_{i,2N_i+2}\theta_1\theta_2^2 + \alpha_{i,2N_i+3}\theta_1^2\theta_2 + \\ &+ \cdots + \alpha_{i,\frac{N_i(N_i+3)}{2}}\theta_1^{N_i-1}\theta_2 \end{aligned}$$

Also in this case, the number of parameters ξ to be identified is significantly impacted by the number of components d of the vector $\theta(t)$, which makes the proposed methodology unfeasible for $d \gg 1$. For $d=2$:

$$\begin{aligned} \xi = n + m + \sum_{i=1}^{n} N_i + \sum_{i=1}^{\omega}\sum_{j=1}^{m} M_{i,j} + \\ + \sum_{i=1}^{n}\gamma_i \frac{N_i(N_i-1)}{2} + \sum_{i=1}^{\omega}\sum_{j=1}^{m}\rho_{i,j}\frac{M_{i,j}(M_{i,j}-1)}{2} \end{aligned}$$

where for $i \in \{1, \cdots, n\}$: $\gamma_i = \begin{cases} 0, N_i \leq 1 \\ 1, N_i > 1 \end{cases}$;

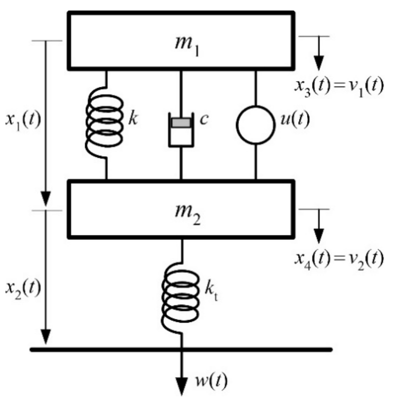
and for $i \in \{1, \cdots, w\}, j \in \{1, \cdots, m\}$, $\rho_{i,j} = \begin{cases} 0, M_{i,j} \leq 1 \\ 1, M_{i,j} > 1 \end{cases}$.

4. Identification of LPV models in a quarter-car suspension system.

This section uses the fourth-order car active suspension model, as per [39] as cited in [40]. This model, in each subsection below, underwent new considerations in relation to the varying parameter, which led to different degrees of nonlinearity and enabled a more detailed and extended analysis of the proposed method.

Figure 2 illustrates the physical model of the quarter-car active suspension. The constant m_1 represents the damped mass of a quarter of the car and m_2 the undamped mass of a wheel and tire assembly. The upper spring, with elastic constant k and the shock absorber, with damping constant c, represent the very suspension of the car. The lower spring, with elasticity constant k_t, refers to the damping generated by the deformation of the tire during the movement of the car on the road. The disturbance $w(t)$ represents the excitation input of the model and consists of a vertical velocity signal due to irregularities found on the road. The input $u(t)$ represents the actuation force produced by the active suspension mechanism. Its purpose is to isolate vibrations in the mass m_1, in addition to providing greater car adhesion to the road.

Figure 2 - Physical quarter-car active suspension model



For the mathematical modeling of the assembly, the state variables of the system can be defined as in [39]:
x_1: distance between the masses m_1 and m_2, from the equilibrium position;
x_2: distance between the wheel axle and its base, also from the equilibrium position;
x_3:vertical speed $v_1(t)$ of the body in relation to the inertial reference;
x_4: vertical speed $v_2(t)$ of the wheel axle in relation to the inertial reference.
Based on the definition of the states presented, it is possible to conclude that:

$$\dot{x}_1 = x_4 - x_3, \tag{20}$$

And also,

$$w(t) = \dot{x}_2 + x_4 \quad \text{or} \quad \dot{x}_2 = w(t) - x_4, \tag{21}$$

Applying Newton's second law to the mass m_1 and considering that there is a linear dependence of the force on the speed in the shock absorber, through the constant c:

$$m_1\dot{x}_3 = kx_1 + c\dot{x}_1 + u. \tag{22}$$

From (20) in (22):

$$\dot{x}_3 = \frac{k}{m_1}x_1 - \frac{c}{m_1}x_3 + \frac{c}{m_1}x_4 + \frac{1}{m_1}u(t). \tag{23}$$

Applying Newton's second law to the mass m_2:

$$m_2\dot{x}_4 = -kx_1 + k_t x_2 + cx_3 - cx_4 - u(t),$$

$$\dot{x}_4 = -\frac{k}{m_2}x_1 + \frac{k_t}{m_2}x_2 + \frac{c}{m_2}x_3 - \frac{c}{m_2}x_4 - \frac{1}{m_2}u(t), \tag{24}$$

In addition, the acceleration of the damped mass m_1 and the state x_2 were considered as output variables [40]. Thus, it is possible to write the mathematical model in state space form, according to:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ \frac{k}{m_1} & 0 & -\frac{c}{m_1} & \frac{c}{m_1} \\ -\frac{k}{m_2} & \frac{k_t}{m_2} & \frac{c}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ -\frac{1}{m_2} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w(t)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{k}{m_1} & 0 & -\frac{c}{m_1} & \frac{c}{m_1} \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{m_1} \\ 0 \end{bmatrix} u(t). \qquad (25)$$

where $y(t) = [y_1\ (t)\ y_2\ (t)]^T$, $y_1\ (t)$ is the acceleration of the mass m_1 and $y_2\ (t)$ the displacement of the mass m_2. The model outputs, according to [40], are related to the acceleration of the damped mass x_3 and the deformation of the tire, x_2.

The nominal values of the parameters adopted [39] were: m_1 = 288,9 kg; m_2 = 28,58 kg; $c = 850\frac{\text{Ns}}{\text{m}}$; $k = 10.000\frac{\text{N}}{\text{m}}$; and $k_t = 155.900\frac{\text{N}}{\text{m}}$. In all cases discussed below, the sampling period T=0,0025s and simulation duration of 2 s were used, which totals 800 periods. In this article, the models were determined with in γ=0 em (10).

In the case of active suspension, considering all measurable states, the control law $u(t) = Kx(t)$ was adopted, where $x(t)$ is the state vector, knowing that the value of K employed was obtained in order to maintain a compromise between passenger comfort and tire adhesion to the road. The value of K presented in [40] and used in this work was:

$$K = 10^3 \times [-9{,}9997 \quad -0{,}0002 \quad +0{,}8325 \quad -0{,}8461]$$

For the simulation of the data to be used in the identification, it was considered that the base of the tire, in contact with the road, is subjected to a disturbance at speed w(t) of the form:

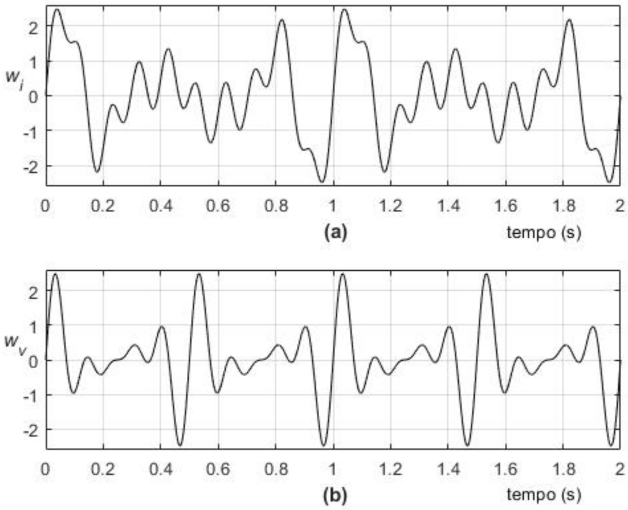
$$w(t) = w_i(t) = 0.9\text{sen}(6\text{p}t) + 0.5\text{sen}(10\text{p}t) + 0.75\text{sen}(8\text{p}t) + 0.6\text{sen}(20\text{p}t), \qquad (26)$$

and for the validation of the models, the same signal as in [40] was used, represented by the equation:

$$w(t) = w_v(t) = 0.6\text{sen}(8\text{p}t) + 0.75\text{sen}(12\text{p}t) + 0.9\text{sen}(16\text{p}t) + 0.5\text{sen}(20\text{p}t), \qquad (27)$$

Figure 3 shows the graphs of the input signals $w(t)$ used for identification and validation.

Figure 3 - Input signals for: (a) identification w_i; (b) validation w_v.



4.1 LTI Models

In this item and the following, it was considered that the mass m_1 (θ) = 288,9 + 100θ and the signal of the varying parameter θ(t) = 0,5t for 0 ≤ t ≤ 2s, in the identification phase. In the validation of the model, another signal was used for the varying parameter, that is, θ(t) = sen(0,5πt) Thus, the mass presented values in the range of 288,9 ≤ m_1 ≤ 388,9, which could correspond to the addition of passengers and luggage in the undamped mass. This consideration is quite conservative in terms of the variation rate of the varying parameter, taking into account a 2-second simulation period. In other words, a variation rate of the parameter slightly higher than what can happen in practice, but respecting that every real physical system is a low-pass system.

VOL.40 Nº1 2023
https://doi.org/10.22491/rmct.v40i1.12121.en

Figures 4 and 5 show, respectively, the system outputs in the cases of passive and active suspension as a function of the excitation with the validation signal and $w_v(t)$ e $m_1(\theta)$.

Figure 4 - Output signals for validating the passive suspension model: (a) y_1; (b) y_2.

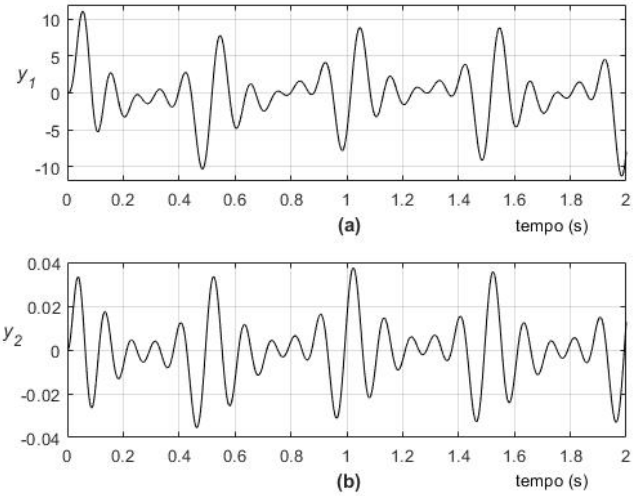


Figure 5 - Output signals for validating the active suspension model: (a) ; (b) .

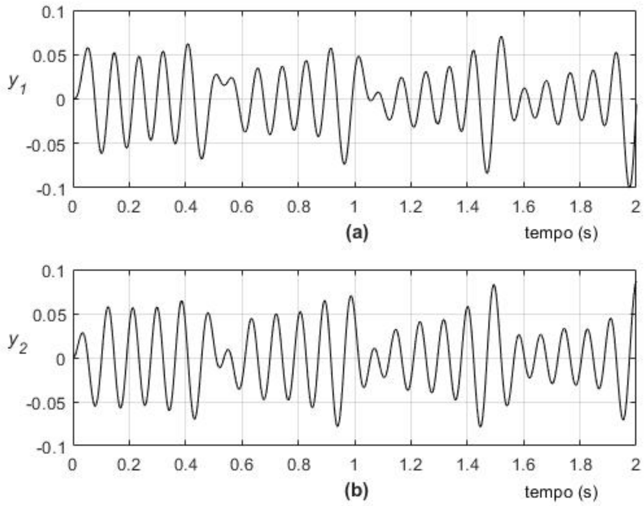


Table 1 shows the characteristics of four LTI models that were adjusted for outputs y_1 and y_2 in the cases of passive and active suspension, remembering that $m_1 = m_1(\theta)$.

It is worth mentioning [40] that the tire deformation x_2, when compared to the outputs y_2 in Figures 4(b) and 5(b), in the active suspension, the excursion was approximately double, while the acceleration y_1 in Figure 5(a) of the damped mass m_1 was of the order of 1% of that occurred for the passive suspension in Figure 4(a).

Table 1 - Characteristics of the identified LTI models.

	Models			
	M1	**M2**	**M3**	**M4**
susp.	passive	passive	active	active
output	y_1	y_2	y_1	y_2
$n/m/r$	4/4/1	4/4/1	2/2/1	2/2/1
N	[0 0 0 0]	[0 0 0 0]	[0 0]	[0 0]
M	[0 0 0 0]	[0 0 0 0]	[0 0]	[0 0]
ξ	8	8	4	4
J_2^a	3.4697e–3	5.3165e–7	1.2436e–3	2.8967e–8
J_∞^a	6.0661e–4	8.3421e–8	1.6489e–4	2.5778e–9
J_2^s	38.274	1.0384e–2	1.5798e–1	1.0063e–5
J_∞^s	4.2958	6.8445e–4	2.4096e–2	8.5410e–7
max|λ|	0.9952	0.9970	0.9994	0.9998

Table 1 shows that the elements of the vectors **N** and **M** in (11) were zero, which corresponds to the adjustment of the LTI models. The values of J_2^a and J_∞^a in this table and in the following ones correspond to the cost of one-step-ahead prediction, according to Figure 1(a), using the validation input $w_v(t)$. It is also worth mentioning that the LTI M2 and M4 models of the outputs y_2, for the passive and active suspension, had satisfactory adjustments, and these outputs are no longer addressed in the next cases. The graphs of M2 and M4 are exactly those shown in Figures 4(b) and 5(b).

Figure 6(a) shows the outputs of the system, in blue, and of the identified LTI M1 model, in red, for passive suspension. Figure 6(b) illustrates the absolute value of the error between these output signals. Figure 7 is the equivalent of Figure 6 for the active suspension case.

Figure 6 - (a) Outputs y_1 of the system (blue) and the LTI M1 model (red) with passive suspension; (b) absolute value of the error between these outputs.

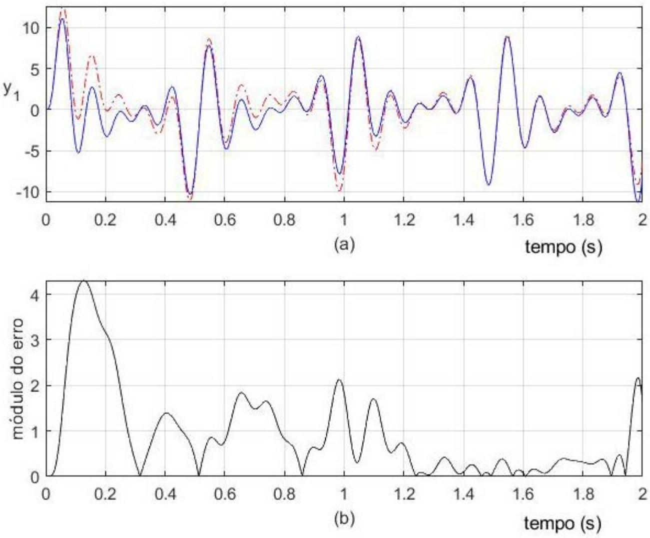
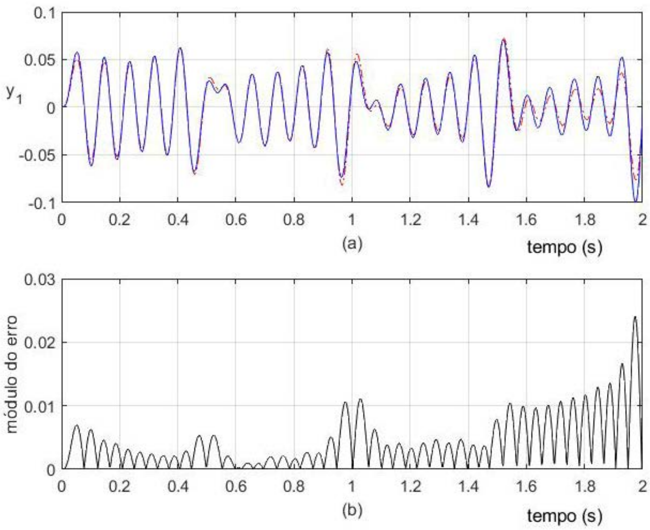


Figure 7- (a) Outputs y_1 of the system (blue) and the LTI M3 model (red) with active suspension; (b) absolute value of the error between these outputs.



It should be noted that the maximum values of the adjustment errors presented in Figures 6(b) and 7(b) correspond to the respective values of J_∞^s in Table 1 for M1 and M3.

4.2 LPV models with one exogenous parameter

In this case, the LPV M5 and M6 models in Table 2 were identified as the best models that reproduced, respectively, the behavior of the outputs y_1 of the car passive and active suspension, considering that the mass m_1 varies over time.

Table 2 - Characteristics of the models identified in 4.2 and 4.3.

	Models			
	M5	**M6**	**M7**	**M8**
susp.	passive	active	passive	passive
$n/m/r$	4/4/1	2/2/1	4/4/1	4/4/1
N	[1 0 0 0]	[1 0]	[0 0 0 0]	[1 0 0 0]
M	[2 0 2 2]	[0 1]	[0 0 0 0]	[1 2 1 1]
ξ	15	6	8	21
J_2^a	5,6473e–4	3,1939e–4	3,2598e–3	3,5658e–4
J_∞^a	1,0453e–4	6,0073e–5	4,0924e–4	5,6716e–5
J_2^s	3,1698	9,1048e–2	39,014	2,6949
J_∞^s	3,4643e–1	9,2476e–3	3,8391	2,4190e–1
max| λ |	0.9966	0.9995	1.0012	0.9980

Comparing the values of J_2^s of M5 and M6 in Table 2 with their corresponding M1 and M3 in Table 1, it can be seen that their adjustments are significantly better, but logically at the expense of the increase in the number of parameters.

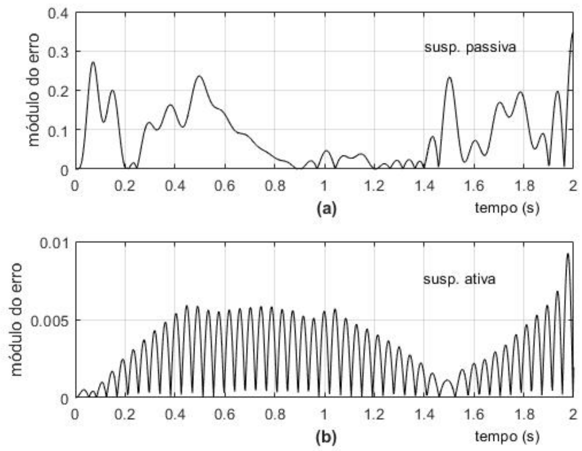
Figure 8 presents the absolute values of the error between the outputs y_1 of the system and the validated models, both for passive suspension and active suspension. It is worth noting that the maximum absolute values of these errors in the graphs of Figure 8 correspond to the values of J_∞^s in Table 2. Comparing the graphs of Figures 6(b) and 8(a), it can be seen that the adjustment error of the LPV model is in the range of 8% of the values presented by the LTI model. Similarly, for the case of active suspension with the output y_1, comparing the graphs of Figures 7(b) and 8(b), it can be seen that the adjustment error of the LPV model presents values lower than 40% of those of the LTI model.

An important characteristic observed in LPV models is that their coefficients depend on θ and, consequently, vary over time. For this reason, it is observed that their eigenvalues also change during the simulation. Thus, since the models are discrete, it is desired that the eigenvalues $\lambda_i = \lambda_i(t)$ meet the following condition:

$$\max_{0 \le t \le 2s} |\lambda_i(t)| < 1.$$

in order to maintain the stability of the model. However, it was found that small exceedance of this limit do not always cause mismatches between the model output and that of the plant. Another aspect that deserves to be mentioned refers to the significant increase in the number of parameters to be adjusted in LPV models. Finally, it was observed that the adjustment of the model is highly dependent on the trajectory of the parameter θ and its speed.

Figure 8 - Absolute error value of the output y_1 for the LPV models: (a) M5 and (b) M6.



4.3 LPV models with two exogenous parameters

In this case, it was considered that the system has two independent exogenous varying parameters, which parameterize the mass m_1 and the spring constant k, as follows:

$$m_1(\theta_1) = 288{,}9 + 100\theta_1,$$
$$k(\theta_2) = 9.000 + 2.000\theta_2,$$

where for $0 \le t \le 2$ s:

$$\theta_1 = \theta_1(t) = 0{,}5t,$$
$$\theta_2 = \theta_2(t) = \mathrm{sen}(0{,}5\pi t).$$

Thus, the parameters m_1 and k varied during the simulation in the following intervals:

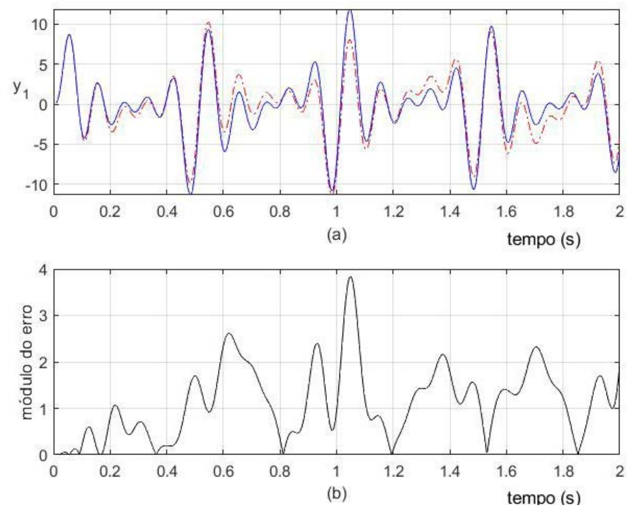
$$288{,}9 \le m_1 \le 388{,}9;$$
$$9.000 \le k \le 11.000.$$

The parameters θ_1 and θ_2 above were used during the identification. In the validation, $\theta_1 = \mathrm{sen}\,(0{,}5\pi t)$ and $\theta_2 = 0{,}5t$.

Table 2 presents the characteristics of two models, M7 and M8, adjusted for the output $y_1(t)$, in the case of passive suspension with the simultaneous variation of m_1 and k. M7 is of the LTI type, while M8 is the LPV for the same data. Figure 9 presents the output estimated by M7, in red, and the corresponding system output. Table 2 shows that the adjustment cost J_2^s of M8 was less than 7% of that presented by M7. It is worth mentioning that the output of M8 coincides with the system output, in blue, in Figure 9.

The identification for active suspension was not performed, since the controller k in [39] leads the closed-loop system to instability with the simultaneous variation of m_1 and k.

Figure 9 - (a) Outputs y_1 of the system (blue) and of the LTI M7 model (red) with passive suspension; (b) absolute value of the error between these outputs.



4.4 LPV models with one endogenous parameter

In this case, the mass m_1 = 288,9 kg was considered, fixed at its nominal value, but the spring model was replaced by a more realistic one [41], which considers the elastic constant k to vary from a certain deformation, as shown in Figure 10. Since the spring deformation is the state x_1, then k becomes dependent on it, that is:

$$k = k(x_1) = \begin{cases} 10^4 \frac{\text{N}}{\text{m}}, & |x_1| \leq 0{,}08 \text{ m} \\ \frac{800+10^5(|x_1|-0{,}08)}{|x_1|} \frac{\text{N}}{\text{m}}, & |x_1| > 0{,}08 \text{ m} \end{cases}$$

and (25) becomes a nonlinear Quasi-LPV model, with $\theta = x_1$.

Figure 10 - Suspension spring force versus deformation.

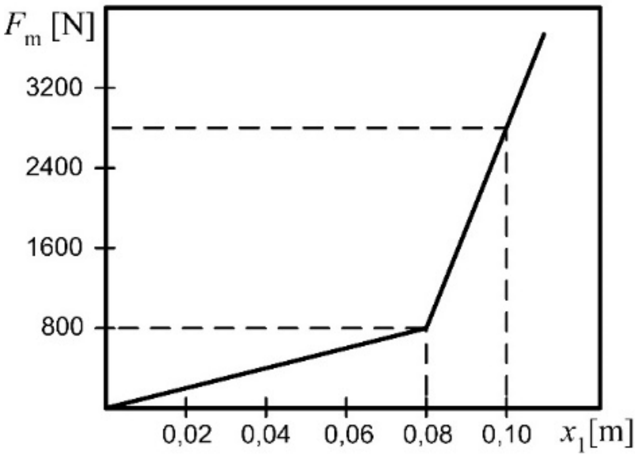


Figure 11 illustrates the behavior of x_1 and k over time for the input signal w_i. When the deformation x_1 exceeds 0.08 m, characterized by the dashed lines in Figure 11(a), the spring constant k becomes variable, as shown in Figure 11(b).

For this case, M9 and M10 from Table 3 were determined, which are distinguished by their parametric structure and the number of parameters. The adjustments of these LPV models were not as good as those identified in the previous cases.

Figure 11 - Temporal evolution for the identification input: (a) state x_1; (b) (b) spring constant.

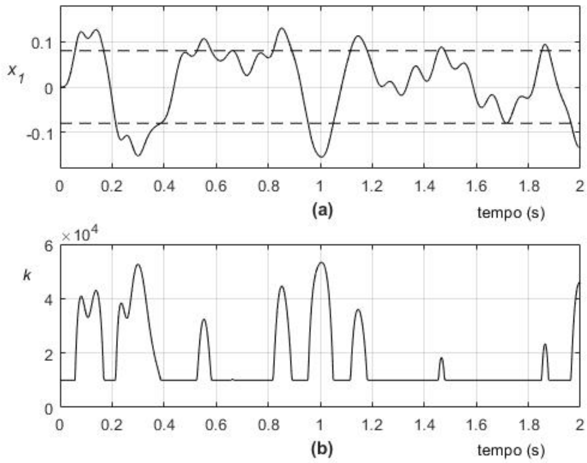


Table 3 - Characteristics of the LPV models identified for passive suspension in 4.4 and 4.5.

	Models			
	M9	**M10**	**M11**	**M12**
$n/m/r$	4/2/3	2/2/1	4/4/1	4/5/0
N	[5 0 3 0]	[2 2]	[0 3 1 2]	[1 0 0 0]
M	[3 4]	[5 2]	[4 2 1 0]	[0 0 3 1 1]
x	21	15	45	24
J_2^a	4.1019	4.7566	8.0199	9.8109
J_∞^a	1.7896	1.8838	2.8412	3.3706
J_2^s	57.719	61.040	52.349	63.038
J_∞^s	6.4872	6.2101	7.5600	8.5038
max|λ|	1.0104	1.0067	1.1402	0.9294

Figure 12 shows the system response, in blue, and that of the validated M9 model, in red. Nevertheless, it was observed that for the one-step-ahead prediction, this model presents a significant improvement in its adjustments, since its estimated responses practically coincide with that of the system, in blue, in this figure. Table 3 confirms this information based on the values of J_2^a presented by M9 and M10. Figure 12(b) illustrates the temporal evolution of the maximum eigenvalue module. It is interesting to note that in some time intervals, this exceeds the unitary value.

It is worth noting that since the state x_1 is the varying parameter itself, θ will be changed with the modification of the identification input to the validation input.

4.5 LPV models with two endogenous parameters

In this case, it was assumed that the suspension is subjected to high speeds. Therefore, a more realistic behavior for the force generated by the shock absorber is one that presents a cubic relationship with the compression or expansion speed. The mathematical model no longer has a linear behavior, but can be transformed into the Quasi-LPV form, as shown below. Equation (22) can be rewritten as:

$$kx_1 + c(\dot{x}_1)^3 + u = m_1\dot{x}_3. \quad (27)$$

From (20):

$$(\dot{x}_1)^3 = (x_4 - x_3)^3 = x_4^3 - 3x_4^2x_3 + 3x_4x_3^2 - x_3^3. \quad (28)$$

From (28) in (27):

$$\dot{x}_3 = \frac{k}{m_1}x_1 - \frac{c}{m_1}(3x_4^2 + x_3^2)x_3 + \frac{c}{m_1}(3x_3^2 + x_4^2)x_4 + \frac{u(t)}{m_1}. \quad (29)$$

Developing (24) in an analogous way, we arrive at:

$$\begin{aligned}\dot{x}_4 = \frac{k}{m_2}x_1 + \frac{k_t}{m_2}x_2 + \frac{c}{m_2}(3x_4^2 + x_3^2)x_3 + \\ -\frac{c}{m_2}(3x_3^2 + x_4^2)x_4 - \frac{u(t)}{m_2}\end{aligned} \quad (30)$$

Using (20), (21), (29) and (30), we arrive at the high-speed active suspension model:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1\\ 0 & 0 & 0 & -1\\ \frac{k}{m_1} & 0 & -\frac{c}{m_1}(3x_4^2 + x_3^2) & \frac{c}{m_1}(3x_3^2 + x_4^2)\\ -\frac{k}{m_2} & \frac{k_t}{m_2} & \frac{c}{m_2}(3x_4^2 + x_3^2) & -\frac{c}{m_2}(3x_3^2 + x_4^2)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} +$$

$$+\begin{bmatrix}0\\ 0\\ \frac{1}{m_1}\\ -\frac{1}{m_2}\end{bmatrix}u(t) + \begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}w(t) \quad (31)$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}\frac{k}{m_1} & 0 & -\frac{c}{m_1}(3x_4^2 + x_3^2) & \frac{c}{m_1}(3x_3^2 + x_4^2)\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}\frac{1}{m_1}\\ 0\end{bmatrix}u(t)$$

From the model in (31), it can be seen that the states x_3 and x_4 compose the matrices A and C of the dynamic, transforming it into a nonlinear Quasi--LPV model.

Figure 12 - (a) Outputs of the system (blue) and the M9 model (red) with passive suspension; (b) Maximum eigenvalue modulus during validation.

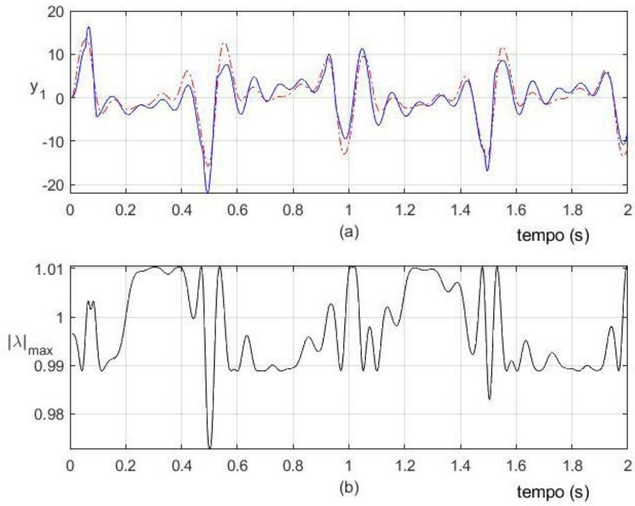


Figure 13 shows the temporal evolution of the states x_3 and x_4 in view of the application of the validation input. In this case, these states will be considered the varying parameters of the LPV model.

Table 3 presents the characteristics of M11 and M12, adjusted for this problem. Although M12 has a worse performance than M11 in terms of the cost J_2^s, it was selected due to the parsimony criterion, as it presents a significantly smaller number of parameters.

https://doi.org/10.22491/rmct.v40i1.12121.en

Even though the adjustment was not perfect, it is still much better than that of the corresponding LTI model, which has $J_2^s = 90,21$. This model was not presented in the tables, but Figure 14 shows the system output curve (blue), of M11 (black) and the output of this LTI model (red).

Figure 13 - Evolution of the state with the validation input: (a) x_3; (b) x_4.

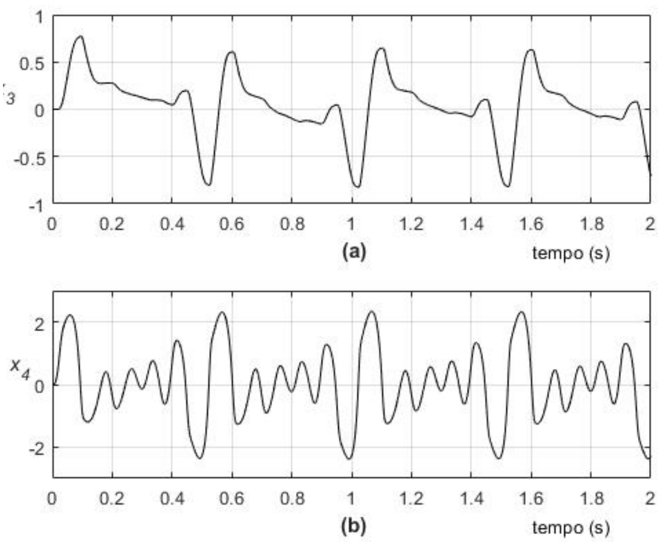
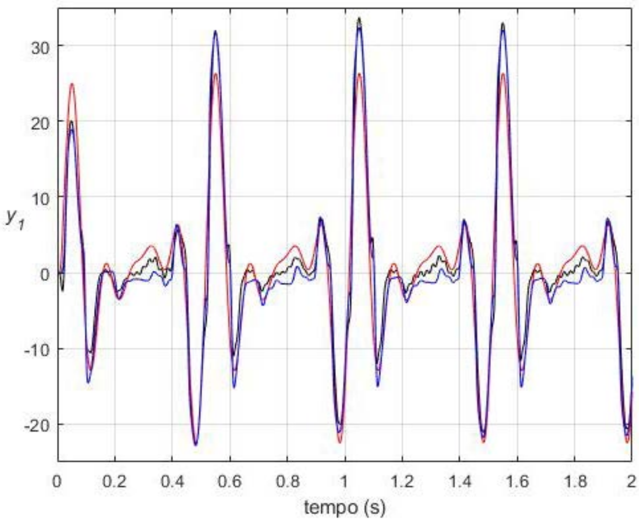


Figure 14 - Outputs of the passive suspension for the system (blue), M11 (black) and LTI (red).



5. Conclusions

In this article, a multivariate method for identifying LPV models with polynomial coefficients was presented. Among the applications of the method, in addition to the identification itself, there is the possibility of approximating nonlinear models by LPV models, with the purpose of applying LPV control techniques. The results were explored through an example related to a car suspension. Several cases were addressed, with multiple endogenous and exogenous varying parameters, seeking to approximate even nonlinear Quasi-LPV systems. Some contributions in relation to the ideas of [24] were implemented, such as the extension of the method to multivariate systems, obtaining the solution from a data batch, the use of polynomials with independent degrees per parameter, the possibility of performing the time lag between the current output and the most recent input in the model, as well as the extension of the method to multiple varying parameters. Although the adjustments of the LPV models were significantly better than those of the LTI models, on the other hand, there was also a large increase in the number of parameters to be determined. It was also observed that the graph of the maximum module of the eigenvalues of the model over time is an important tool for analyzing the quality of the identified model and its chosen parametric structure. It is worth remembering that the one-step-ahead prediction showed excellent results for all the models presented, although there is a dependence on the output measurements of the system in real time. Finally, it was verified, mainly in the approximation of nonlinear systems, the strong dependence of the adjustment of the model on the trajectory of the variable parameter and its variation rate.

Acknowledgments

This article was prepared with the support from the Coordination for the Improvement of Higher Education Personnel - Brazil (CAPES) - Financing Code 001.

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Appendix A - Model coefficients

Table A1 - Coefficients of the identified models.

	M1	M2	M3	M4
1	–3.8899	–3.8875	–1.9650	–1.9657
2	5.7064	5.6979	9.9883e–1	9.9966e–1
3	–3.7426	–3.7329	2.0657e–4	2.4858e–3
4	9.2613e–1	9.2258e–1	1.8604e–4	–2.4850e–3
5	–1.4126e–2	2.4678e–3		
6	1.2122e–1	–7.2040e–3		
7	–1.9795e–1	7.0104e–3		
8	9.0897e–2	–2.2742e–3		

	M5	M6	M7	M8
1	–3.8854	–1.9650	–3.9020	–3.8902
2	–1.6614e-6	–4.2788e–5	5.7371	1.5265e–6
3	5.6915	9.9900e–1	–3.7683	8.0307e–7
4	–3.7266	1.9014e–4	9.3315e–1	5.7055
5	9.2050e–1	2.6350e–4	1.3017e–1	–3.7401
6	4.7693e–2	–1.1792e–4	–3.1863e–1	9.2488e–1
7	–1.0689e–2		2.4856e–1	3.7301e–2
8	2.3814e–3		–6.0094e–2	1.1522e–2
9	–4.4441e–2			1.1000e–2
10	–5.1441e–2			–3.8458e–2
11	3.1428e–2			–1.0157e–2
12	–7.0396e–3			–4.3035e–5
13	4.8188e–2			–3.5744e–2
14	–2.0730e–2			–4.7985e–6
15	4.6544e–3			–5.4108e–7
16				–3.3115e–2
17				–1.3655e–2
18				3.8937e–2
19				3.4275e–2
20				1.2319e–2
21				–1.4199e–2

	M9	M10	M11	M12
1	–2.1451	–1.8308	–2.6426	–2.8394
2	1.9944e-1	8.0398e–2	3.0743	–1.9056e–2
3	–3.2760	–4.5524	–1.9459e–1	2.1094e–3
4	–24.188	8.3022–1	–1.2791e–2	3.4121
5	28.608	–5.1005e–2	–2.7967e–2	–2.0704
6	536.92	4.9448	–1.2417e–1	5.2334e–1
7	1.4962	8.8089e–1	–4.5444e–2	–255.01
8	–3.3536e–1	–9.3699e–1	3.4387e–3	1024.2
9	–1.2673e–1	–5.1392e–1	9.8325e–2	–1548.7
10	2.9411	70.379	–1.2912e–2	-5.6339
11	10.771	–850.63	3.6693e–2	1.1043e–1
12	–1.5032e–2	–4000.2	–1.6806	–1.0481e–1
13	8.7969e–1	–8.7297e–1	4.6561e–1	1.8862e–1
14	–5.7447e–1	6.8925e–1	2.1312e–1	2.5113e–2
15	15.912	16.264	3.3240e–1	7.4621e–3
16	144.42		–2.3406e–1	–1.0500e–1
17	–8.5403e–1		–2.3056e–2	–4.1403e–2
18	7.6552e–1		–1.0547e–1	9.2937e–2
19	–5.0067		2.7712e–2	1045.4
20	–176.37		–6.0313e–2	11.942
21	–565.46		–8.8240	–5.4097e–1
22			–3.6792	-265.75
23			3.8992	-6.4258
24			–1.4866e–1	3.4988e–1
25			–3.3907e–1	
26			–9.5981e–2	
27			7.8418e–1	
28			2.0652e–2	
29			–3.9837e–2	
30			–2.6922	
31			–1.1238e–1	
32			2.9413e–1	
33			2.7791e–1	
34			–5.6398e–1	
35			5.4148e–1	
36			26.288	

	M9	M10	M11	M12
37			6.5098	
38			–3.2759	
39			1.4140e–1	
40			–6.5476e–1	
41			1.9229	
42			–27.293	
43			–2.5943	
44			–1.2012e–1	
45			10.132	

The coefficients of the models presented in Table A1 are arranged in the order shown in (7), that is, a_1, a_2, a_3, ..., b_1, b_2, b_3, ... As each coefficient has several parameters, they are in the sequence shown in item 3.2.